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Properties and Characterizations of k-Continuous Functions and k-Open Sets in Topological Spaces

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ABSTRACT

Topology, a branch of mathematics focused on understanding the fundamental properties of spaces, has experienced a new dimension with the introduction of k -open sets. This exploration delves into the realm of k -topology, unveiling concepts such as k -interior, k -closure, k -limit points, k -continuous functions, and more. Through rigorous definitions, illustrative examples, and a web of theorems, the study navigates the intricate relationships between k -open sets and traditional topology. We analyze the interplay of k -functions, investigate the properties of k -homeomorphisms, and discuss the connections between k -totally continuous and k -contra-continuous functions. By merging established topology with these novel notions, this study sheds light on the intricate fabric of topological spaces (X, τ)(briefly top. sp. X) from a unique perspective.

Keywords: Topological Spaces, Homeomorphisms, Contra-Continuous Functions, Totally Continuous

INTRODUCTION

Topology is a branch of mathematics that delves into the fundamental properties and relationships between spaces, emphasizing concepts like continuity, convergence, and openness. Traditional topology has been instrumental in understanding the structure of spaces through the lens of open sets, continuous functions, and various topological properties. However, recent developments have extended this realm by introducing the notion of " k -open sets," leading to the emergence of a new set of concepts and properties that provide novel insights into top. sp.

In this exploration, we embark on a journey through the world of k -open sets and their associated concepts. These concepts, ranging from k -interior and k -closure to k -limit points, k -derivatives, and more, offer a fresh perspective on how subsets interact within topological spaces. We investigate the interplay between k -open and k -closed sets, delve into the properties of k -continuous functions, and examine the intricacies of k -totally continuous and k -contra-continuous functions. Moreover, the concept of k -homeomorphism introduces a fascinating connection between k -open sets and function properties.

Throughout this study, we provide rigorous definitions, explore theorems, and present illustrative examples to aid in understanding these intricate concepts. By combining established topology with these novel notions, we deepen our understanding of top. sp. and pave the way for new perspectives in the field of mathematics.

Preliminaries

Before delving into the specifics of k -open sets and related concepts, let's establish a preliminary overview of the foundational principles of topology. We begin by reviewing the basics of open and closed sets, and the topology of space. Understanding the concept of continuity is crucial, as it serves as the bridge between the properties of spaces and functions. We also touch upon important notions like compactness, convergence, and the topological properties that emerge from the interplay of open sets. With this foundational knowledge in place, we transition to the main focus of our study k -open sets. We define k -open sets and examine their properties, drawing parallels and distinctions from traditional open sets. This leads us to explore k -interior, k -closure, k -limit points, and other related concepts. We delve into the relationships between these concepts and demonstrate their implications through illustrative examples. As we progress, we introduce k -cont. funs and investigate their properties. We explore the connections between k -continuous, k -irresolute, k -totally continuous, and k-contra-cont. funs, revealing intriguing relationships and counterexamples. The notion of k-homeomorphism adds another layer of depth to our study, showcasing how k -open sets and function properties interact. Overall, this preliminary overview sets the stage for a comprehensive exploration of k -open sets and their role in understanding top. sp. Armed with this foundation, we embark on a journey through the intricacies and implications of these concepts, enriching our understanding of topology from a fresh perspective. Now, we present a novel category of open sets known as k-open sets. Through the utilization of these k-open sets, we establish the concepts of k-interior, k-closure, k-limit points, k-derived, k-border, k-frontier, and k-exterior, and delve into their topological characteristics.

Definition 2.1: Let X be a top. sp. A subset $\xi \subseteq X$ is called k-open if for every set $\emptyset \neq U \subseteq X$ with $U \neq X$ and $U \in \tau$ (U is an open set), $\xi \subseteq cl(\xi \cup int(U))$. The complement of k-open set ξ in the top. sp. X is called k-closed. In other words, a set B is k-closed iff$B$ is k-open. The family of all k-open sets of the top. sp. X is denoted by τ^k.

Properties of k-open sets and k-closed sets 2.2:

The concept of k -open sets introduces a different perspective on openness within a topology. Here are some properties of k-open sets:

1. Empty Set and Whole Space: $\emptyset$ and X are k -open.
2. Finite Intersection: The k -open property remains preserved when taking the intersection of a finite collection of k -open is k -open.
3. Arbitrary Union: The union of arbitrary k -open is not guaranteed to be k -open. Unlike open sets in the standard topology, k -open do not necessarily preserve openness under arbitrary unions.

4. Closure of k -open Set: The closure of k-open need not be k-open. This is in contrast to open, where the closure of an open is still open.
5. Interior of k-Closed Set: The interior of k-closed need not be k-closed. In the standard topology, the interior of closed set is open.
6. Boundary of k -Open Set: The boundary of k-open is contained within set itself. In traditional topology, the boundary of an open might have points both inside and outside the set.
7. Frontier of k -Open Set: The frontier (also known as the boundary) of k-open is contained within set itself.
8. Derived Set of k-Open Set: The derived set of k -open might not be k-open. In standard topology, the derived set of an open is generally not open.
9. k -Open Sets vs. Open Sets: In the general case, k -open sets are a broader class than open . This means that every open is k -open set, However, the opposite may not always hold true.
10. Relationship to Topology: The properties of k -open sets are closely tied to the specific definition used in this concept. They may not always match the intuitive properties of open sets in the standard topology.

These properties highlight the distinctive nature of k -open sets compared to open sets in traditional topology. The concept provides an alternative way to define openness within a space, but it can lead to different topological behaviors.

Examples 2.3: Real Numbers with the k-topology. We consider the real numbers $\mathbb{R}$ with k-top., A set is considered open if it encompasses its own accumulation points. In this case, let's compare open in standard topology with k -open.

1. Open Intervals in Standard Top.: The interval $(0, 1)$ is open set in the standard topology. However, in the k -topology, this isn't k -open, since its complement $\left([-\infty, 0] \cup [1, \infty)\right)$ isn't k -closed. The boundary points 0 and 1 are not contained in the complement's closure.
2. Closed Sets in Standard Top.: The interval $[0, 1]$ is closed set in standard topology. But, in k -top., it's not k -closed since its interior is not k -closed.
3. Finite Intersection: The intersection of finitely many k -open in k -topology remains k -open. This is similar to the standard topology's property of finite intersection of open sets being open.

Example 2.4: Discrete Topology. Consider a set X with discrete top., where every subset is open. Let's examine properties of k-open sets in this context.

1. Open Sets in Standard Topology: Since every subset is open, every subset is also k -open, because the closure of any subset is the subset itself.

2. Union of k -Open Sets: Union of arbitrary k -open in the discrete top. is not necessarily k -open. For instance, if we take a union of two disjoint sets, their closure may not cover the union.
3. Intersection of k-Closed Sets: In the discrete topology, all sets are closed. However, the intersection of k -closed sets need not be k -closed.

Example 2.5: Finite Space. Consider a finite set X with the discrete topology. Let's see how k -open sets behave here.

1. Finite Sets in Standard Topology: All finite subsets are open in the standard topology, and therefore, all finite subsets are k -open as well.
2. Infinite Union: The infinite union of k -open sets in this case is not guaranteed to be k -open. The k-closure may not cover the entire union.
3. Boundary and Frontier: The boundary and frontier of k-open set will be set itself, similar to the discrete topology case.

These examples illustrate how k -open sets might behave differently from standard open in various topologies. The k -top. introduces different perspective on openness and closedness, leading to distinct topological properties.

Theorem 2.6: n any given topological space X, each open set is categorized as a k-open set. However, the reverse statement does not hold.

Proof: Consider an open set ξ in the top. sp. X. To prove ξ is k -open set, i.e., for every non-empty set $\emptyset \neq U \subseteq X$ $(U \neq X)$ and $U \in \tau, \xi \subseteq Cl(\xi \cup Int(U))$. Let $U \neq \emptyset$ in X such that $U \neq X$, $U \in \tau$ (i.e., U is an open). Since ξ open, by definition, $\xi \subseteq U$. Now, To show that $\xi \subseteq Cl(\xi \cup Int(U))$.Notice that $Int(U)$ is subset of U, and therefore $\xi \cup Int(U)$ also subset of U.Then, by the property of closure (Cl), we know that $\xi \subseteq Cl(\xi \cup Int(U))$, since $Cl(\xi \cup Int(U))$ is smallest closed containing $\xi \cup Int(U)$, and $\xi \cup Int(U)$ is subset of U.Therefore, we have shown that for every $\emptyset \neq U \subseteq X$, $(U \neq X)$ and $U \in \tau, \xi \subseteq Cl(\xi \cup Int(U))$, which means ξ is a k-open set.

Converse: Its isn't necessarily true. That is, not every k-open is open. The definitions of openness and k -openness are different, and a set being k -open depends on the specific conditions stated in its definition $\left(\xi \subseteq Cl(\xi \cup Int(U))\right)$. This means that there can be k -open sets that are not open according to the standard topology.

In summary, every open set in a top. sp. is a k -open set, but not every k -open set is necessarily an open set. The concepts of open sets and k -open sets are related but distinct, and their properties can differ in certain contexts.

Now, we rewrite the theorem 2.6 as:

Theorem 2.7: Every open set in any top. sp. X is a k-open set iff every k-open set is open.

Proof: Direction 1: Every open is k-open. We showed if ξ is an open in a top. sp. X, then for every $\emptyset \neq U \subseteq X$, $(U \neq X)$ and $U \in \tau, \xi \subseteq Cl(\xi \cup Int(U))$, satisfying the definition of k-openness.

Direction 2: Every k-open is open. Let ξ be a k-open in the top. sp. X. For every $\emptyset \neq U \subseteq X$, $(U \neq X)$ and $U \in \tau$, we have $\xi \subseteq Cl(\xi \cup Int(U))$.Let's consider $U = \xi$. Since $U \in \tau$ (by the definition of top. sp., every open is in the topology), we have $\xi \subseteq Cl(\xi \cup Int(A))$.Now, notice that $\xi \subseteq Int(\xi)$ (since every set is a subset of its interior). Therefore, $\xi \subseteq Cl(\xi \cup Int(\xi)) = Cl(\xi)$ (since $\xi \subseteq Int(\xi)$).Since $\xi \subseteq Cl(\xi)$, and $Cl(\xi)$ is closed set containing ξ, by definition of open, ξ must be an open. Therefore, every k-open is open. So, if every open is k-open set and every k-open set is open, then openness and k-openness are equivalent concepts, meaning that a set is open iffits k-open.

Definition 2.8 Let X be top. Sp., and let ξ be subset of X. The k-interior of ξ is defined as the amalgamation of all k-open within X, denoted as $Int_k(\xi)$. It's evident that $Int_k(\xi)$ is k-open set for any subset ξ of X .

Example 2.9: Consider the top. sp. X ,where $X = \{a, b, c, d\}$ and $\tau = \{\emptyset, X, \{a\}, \{a, b\}, \{a, c\}, \{a, b, c\}\}$. the k-interior $Int_k(\xi)$ for subset $\xi \subseteq X$. Let's take $\xi = \{a, b\}$ as an example subset. $Int_k(\{a, b\}) = \{\emptyset\} \cup \{a\} \cup \{a, b\} \cup \{a, b, c\} = \{\emptyset, a, b, c\}$

Proposition 2.10: Let X be a top. sp. and let $\xi \subseteq B \subseteq X$. Then:

1. $Int_k(\xi) \subseteq Int_k(B)$.
2. $Int_k(\xi) \subseteq \xi$.
3. ξ is k-open iff $\xi = Int_k(\xi)$.

Definition 2.11: Assume X is a topological space and let ξ be a subset of X. The k-closure of ξ, denoted as $Cl_k(\xi)$, is determined as the intersection of all k-closed sets within X that encompass ξ. It's evident that $Cl_k(\xi)$ qualifies as a k-closed set for any subset ξ of X.

Example 2.12: Consider the top. sp. X where $X = \{a, b, c, d\}$ and $\tau = \{\emptyset, X, \{a\}, \{a, b\}, \{a, c\}, \{a, b, c\}\}$. The k-closure $Cl_k(\xi)$ for subset $\xi \subseteq X$. Let's take $\xi = \{a, b\}$ as an example subset. $Cl_k(\{a, b\}) = X \cap \{a, b\} \cap \{a, b, c\} \cap \{a, b, c, d\} = \{a, b\}$. As we can see, $Cl_k(\{a, b\})$ is a k-closed and is indeed the intersection of k-closed containing $\{a, b\}$. Under these circumstances, the k-closure is $\{a, b\}$. The k-closure $Cl_k(\xi)$ captures the concept of closedness considering k-closed sets, and it is not necessarily the same as the usual closure defined by standard topology. The k-closure $Cl_k(\xi)$ ensures that the intersection of k-closed containing ξ is itself a k-closed set.

Definition 2.13. Let X as a top sp, and let $\xi \subseteq X$. An element $x \in X$ is designated as a k-limit point of ξ if it fulfills the following condition: for any G in the k-topology τ^k, if x is an element of G, then the intersection of G with the set $\xi\{x\}$ is non-empty.

Example 2.14: Consider the top. sp. X where $X = \{a, b, c, d\}$ and $\tau = \{\emptyset, X, \{a\}, \{a, b\}, \{a, c\}, \{a, b, c\}\}$. Let's find the k-limit points of set $\xi \subseteq X$. Let's take $\xi = \{a, b\}$ as an example subset. Therefore, the k-limit points of the set $\xi = \{a, b\}$ in this topology are $\{a, b\}$.

Example 2.15: Let X be set of real numbers with the usual top denoted by τ. In other words, τ comprises the set of all open intervals (a, b), where both "a" and "b" are real numbers. Consider the set $\xi = (1, 3) \cup \{4\}$. The k-interior of ξ, denoted by $Int_k(\xi) = (1, 2) \cup (1, 3) \cup (2, 3) \cup (4, 5)$

Definition 2.16: Let X be top. sp., and let ξ be a subset of X. Within this context, a point $x \in X$ is defined as k-limit point of ξ if it adheres to the following condition: for every k-open set G (denoted by τ^k), if x is an element of G, then the intersection of G and the set ξ excluding the point x (denoted as $\xi\{x\}$) is not empty. The collection of all k-limit points of the set ξ is referred to as the k-derived set of ξ, denoted as $D_k(\xi)$.

Example 2.17: Consider the top. sp. X, In this context, X represents the set of real numbers equipped with the standard top., denoted as τ. (open intervals). Let's work with a specific set ξ and its k-limit points and k-derived set are Let $\xi = (1, 3) \cup \{4\}$. The k-derived set $D_k(\xi)$represents the collection of all accumulation points of ξ with respect to the parameter k.$D_k(\xi) = \{1, 2, 3, 4\}$

Theorem 2.18: If x is not k- accumulation point of ξ, then there exists a k-open set G in X such that $x \in G$ and $G \cap (\xi\{x\}) = \emptyset$.

Proof: Let's prove this statement by considering both cases: when x is not a k- accumulation point of ξ, and when x is a k- accumulation point of ξ. Let x is not a k- accumulation point of ξ, it means that there exists a k-open G_x in X such that $x \in G_x$, but the intersection $G_x \cap (\xi\{x\})$ is empty. This is because x not being a k- accumulation point implies that there is a k-open containing x, but that set doesn't have a non-empty intersection with $(\xi\{x\})$. Suppose x is a k- accumulation point of ξ: then for every k-open G containing x, the intersection $G \cap (\xi\{x\}) \neq \emptyset$. This means that there is no k-open that includes x and has an empty intersection with $(A\{x\})$.

Example 2.19:

1. Let's consider the set $\xi = (1, 3) \cup \{4\}$ and point $x = 2$. We've already seen that $x = 2$ is a k- accumulation point of ξ. However, if we consider a k-open $G = (1, 3)$, it contains $x = 2$ but has an empty intersection with $(\xi\{2\})$, which is $\{(1, 3), 4\}\{2\} = \{(1, 3), 4\}$. This example shows that $x = 2$ is not k- accumulation point of ξ.

2. Let's consider the set $\xi = (1, 3) \cup \{4\}$ and point $x = 1$. We've already seen that $x = 1$ is k- accumulation point of ξ. For any k-open G containing 1, the intersection $G \cap \big((1, 3) \cup \{4\} \{1\}\big) \neq \emptyset$. This is the characteristic of a k- accumulation point.

In summary, if x is not k- accumulation point of ξ, there exists a k-open G containing x such that $G \cap (\xi\{x\}) = \emptyset$. Conversely, if x is a k-limit point of ξ, for every k-open set G containing x, the intersection $G \cap (\xi\{x\}) \neq \emptyset$. This showcases the behavior of k-accumulation points in relation to k-open and their intersections with ξ.

Theorem 2.20: Let X be a top. sp., and let ξ be subset of X. The following statements are equivalent:

1. $(\forall G \in \tau^k)(x \in G \Rightarrow \xi \cap G \neq \emptyset)$.
2. $x \in Cl_k(\xi)$.

Proof: To prove that statements 1 and 2 are equivalent, we will prove the two implications:

Implication (1) ⇒ (2): Assume that for every k-open set G in τ^k, if $x \in G$, then $\xi \cap G \neq \emptyset$. We want to show that $x \in Cl_k(\xi)$. By definition, $x \in Cl_k(\xi)$ iff for every k-open set G in τ^k containing x, $G \cap (\xi\{x\}) \neq \emptyset$. Let's take any k-open set G containing x. According to our assumption (1), $\xi \cap G \neq \emptyset$. This implies that there is element y in both ξ and G. If $y \neq x$, then $y \in G \cap (\xi\{x\}) \neq \emptyset$. If $y = x$, then since $\xi \cap G \neq \emptyset$, there must be an element $x \neq z$ in ξ and is in G. Therefore, $z \in G \cap (\xi\{x\}) \neq \emptyset$. In either case, we prove that for every k-open G containing $x, G \cap (\xi\{x\}) \neq \emptyset$, which implies that $x \in Cl_k(\xi)$.

Implication (2) ⇒ (1): Assume that $x \in Cl_k(\xi)$. We prove that for every k-open G in τ^kcontaining x, $\xi \cap G \neq \emptyset$. By definition, $x \in Cl_k(\xi)$ means that for every k-open G in τ^k containing x, the intersection $G \cap (\xi\{x\}) \neq \emptyset$. Let G be any k-open set in τ^k containing x. If $G \cap (\xi\{x\}) \neq \emptyset$, then we are done, and $\xi \cap G \neq \emptyset$. If $G \cap (\xi\{x\}) = \emptyset$, it means that there is no element of ξ in G other than x. However, since $x \in Cl_k(\xi)$, every k-open set containing x must have a non-empty intersection with ξ (excluding x). This implies that there must be an element of ξ other than x in G, which leads to a contradiction. Since the assumption $G \cap (\xi\{x\}) = \emptyset$ leads to a contradiction, it must be the case that $G \cap (\xi\{x\}) \neq \emptyset$, and thus $\xi \cap G \neq \emptyset$. Therefore, the two statements are equivalent, and the theorem is proved.

Theorem 2.21: Let X be a top. sp., and let $\xi \subseteq B \subseteq X$. The following statements hold:

1. $Cl_k(\xi) = \xi \cup D_k(\xi)$.
2. ξ is k-closed $D_k(\xi) \subseteq \xi$.
3. $D_k(\xi) \subseteq D_k(\xi)$.
4. $D_k(\xi) \subseteq D(\xi)$.
5. $Cl_k(\xi) \subseteq Cl(\xi)$.

Theorem 2.22: Let τ_1 and τ_2 be topologies on X such that $(\tau_1)^k \subseteq (\tau_2)^k$. For any subset ξ of X, every k- accumulation point of ξ with respect to τ_2 is a k - accumulation point of ξ with respect to τ_1.

Proof: Let's prove this theorem by considering a k -limit point x of ξ with respect to τ_2. According to Definition 2.13, for every k-open set G in $(\tau_2)^k$ such that $x \in G$, the intersection $G \cap (\xi\{x\}) \neq \emptyset$. Since $(\tau_1)^k \subseteq (\tau_2)^k$, every k-open set in $(\tau_1)^k$ is also a k-open set in $(\tau_2)^k$. Therefore, the k-open sets from τ_1 are also k-open sets from τ_2. Now, consider any k-open set G in $(\tau_1)^k$ containing x. Since G is a k-open set in $(\tau_2)^k$as well, according to the definition of a k-limit point with respect to τ_2, $G \cap (A\{x\}) \neq \emptyset$. This suggests that x serves as a k-limit point of ξ under τ_1 as well. Put differently, any k-limit point of ξ under τ_2 is also a k-limit point of ξ under τ_1.based on the given condition that $(\tau_1)^k \subseteq (\tau_2)^k$.

Example 2.23. $X = \{a, b, c\}, \tau_1 = \{\emptyset, X, \{a\}\}$ and $\tau_2 = \{\emptyset, X, \{a\}, \{a, b\}$.To determine whether $(\tau_1)^k$ is subset of $(\tau_2)^k$. Given: $X = \{a, b, c\}, \tau_1 = \{\emptyset, X, \{a\}\}, \tau_2 = \{\emptyset, X, \{a\}, \{a, b\}\}$,$(\tau_1)^k = \{\emptyset, X, \{a\}\}$,$(\tau_2)^k = \{\emptyset, X, \{a\}, \{a, b\}\}$. Now, let's compare $(\tau_1)^k$ and $(\tau_2)^k$. As we can see, $(\tau_1)^k \nsubseteq (\tau_2)^k$, since $\{a\}$ is not present in $(\tau_2)^k$. Therefore, the given statement $(\tau_1)^k \subseteq (\tau_2)^k$ is not true in this case.

Example2.24: Consider the set $X = \{a, b, c\}$ and the following topologies: $\tau 1 = \{\emptyset, X, \{a\}, \{b\}\}, \tau 2 = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$.then $(\tau_1)^k = \emptyset, X, \{a\}, \{b\}, \{a\}$, and $\{b\}$,and $(\tau_2)^k = \emptyset, X, \{a\}, \{b\}, \{a, b\}, \{a\}$, and $\{b\}$.Therefore, Under these circumstances, the assertion $(\tau_1)^k \subseteq (\tau_2)^k$ is true.

Theorem 2.25.

Suppose X is a top. sp., and let ξ and B be subsets of X. If ξ is k-closed, then the k-closure of the intersection of ξ and B is contained within the intersection of ξ and the k-closure of B. i.e., $Cl_k(\xi \cap B) \subseteq \xi \cap Cl_k(B)$

Proof: Assume that ξ is k-closed, which means $\xi = Cl_k(\xi)$. We want to show that $Cl_k(\xi \cap B) \subseteq \xi \cap Cl_k(B)$.First, consider any point x in $Cl_k(\xi \cap B)$. This means that x is in the intersection of all k-closed sets containing $\xi \cap B$.Since ξ is k-closed, ξ is itself a k-closed set containing ξ. So, for every k-open set G containing x, we have $G \cap (\xi\{x\}) \neq \emptyset$. Also, since x is in $Cl_k(\xi \cap B)$, x is in the intersection of all k-closed sets containing $\xi \cap B$. This implies that x is in the intersection of all k-closed sets containing B, because $\xi \cap B$ is a subset of B.Therefore, $x \in Cl_k(B)$, which means $x \in \xi \cap Cl_k(B)$.Now we have shown that every point $x \in Cl_k(\xi \cap B)$ is also in $\xi \cap Cl_k(B)$.Hence, $Cl_k(\xi \cap B) \subseteq \xi \cap Cl_k(B)$, as required.

Lemma 2.26: Let X be a top. sp., and let ξ be a subset of X. Then ξ is k-open iff there exists an open U in X such that $\xi \subseteq U \subseteq Cl(\xi)$.

Proof: ξ is k-open ⇒ There exists $U: \xi \subseteq U \subseteq Cl(\xi)$). Assume that ξ is k-open. This means that for every non-empty open set G in X, if G contains ξ, then $\xi \subseteq Cl(\xi \cup int(G))$according to the definition of k-open sets. Let $U = Cl(\xi \cup int(G))$. Since G is open and G contains ξ, U is also an open set. $\xi \subseteq U$. By construction, $U = Cl(\xi \cup int(G))$, and since $\xi \subseteq \xi \cup int(G), \xi$ is a subset of $U. U \subseteq Cl(\xi)$. Since $U = Cl(\xi \cup int(G)), U$ is a closed set that contains ξ. This means that every point in U is either in ξ or a limit point of ξ. Therefore, $U \subseteq Cl(\xi)$. Hence, in this direction, we have shown that if ξ is k-open, then there exists an open set U in X such that $\xi \subseteq U \subseteq Cl(\xi)$. Backward Direction (There exists $U: \xi \subseteq U \subseteq Cl(\xi) \Rightarrow \xi$ is k-open): Assume that there exists an open set U in X such that $\xi \subseteq U \subseteq Cl(\xi)$. This implies that every point in U is either in ξ or a limit point of ξ. Let G be a non-empty open set in X that contains ξ. According to our assumption, $G \cup int(G)$ is an open set that contains A. Since $G \subseteq G \cup int(G)$, every point in G is also in U. This means that for every point x in G, x is either in ξ or a limit point of ξ. According to Definition 2.1 of k-open, for every non-empty open set G that contains $\xi, \xi \subseteq Cl(\xi \cup int(G))$, which is the same as $\xi \subseteq Cl(\xi \cup G)$. Since G is open and contains ξ, this condition is satisfied for k-openness. Hence, in this direction, We've demonstrated that if an open set U exists within the space X such that $\xi \subseteq U \subseteq Cl(\xi)$, then ξ is k-open. Combining both directions, we conclude that ξ is k-Open iff there's open U within X such that $\xi \subseteq U \subseteq Cl(\xi)$.

Lemma 2.27: The k-openness of a set obtained by intersecting an open set and a k-open set is preserved.

Proof: Let U be an open set and V be a k-open in a top. sp. X.To prove that $U \cap V$ is a k-open set. According to Definition 2.1 of k-open sets, we need to show that for every $\emptyset \neq G$ in X $(G \neq X)$ that is open, $(U \cap V) \subseteq Cl((U \cap V) \cup int(G))$. Let G be a non-empty open set in X $(G \neq X)$. Now, let's consider the set $(U \cap V) \cup int(G)$. Since $int(G) \subseteq G$, we have $(U \cap V) \cup int(G) \subseteq (U \cap V) \cup G$. Now, let's look at the closure of $(U \cap V) \cup G$. Since U is open, $U \cap G$ is also open. Thus, $Cl(U \cap G) \subseteq Cl(G)$. Since V is k-open, $(U \cap V) \subseteq Cl((U \cap V) \cup int(G)) = Cl(U \cap G) \subseteq Cl(G)$. This shows that $(U \cap V) \cup int(G) \subseteq Cl((U \cap V) \cup int(G))$. Since this condition holds for every open $\emptyset \neq G$ in X $(G \neq X)$, we can conclude that $U \cap V$ is a k-open , according to Definition 2.1. Therefore, the intersection of an open set U and a k-open set V, which is $U \cap V$, is indeed a k-open set.

Example 2.28: Given $X = \{a, b, c\}, \tau_1 = \{\emptyset, X, \{a\}\}, \tau_2 = \{\emptyset, X, \{a\}, \{a, b\}\}$. $(\tau_1)^k = \{\emptyset, X, \{a\}, \{a\}\}$. . $(\tau_2)^k = \{\emptyset, X, \{a\}, \{a, b\}, \{a\}, \{a, b\}\}$These are the families of k-open sets for the respective topologies $(\tau_1)^k$ and $(\tau_2)^k$.

Theorem 2.29: If τ is the indiscrete (trivial) top. or the discrete top. on X, then τ^k (the family of all k-open sets) is also indiscrete (trivial) top. or the discrete top. on X, respectively.

Proof: Case 1: τ is the indiscrete (trivial) topology: In indiscrete top., the only open sets are $\emptyset$ and X. According to Definition 2.1, the k-open sets are subsets ξ of X for which $\xi \subseteq Cl(\xi \cup int(G))$

for every non-empty set G $(G \neq X)$ that is open. Since the only $\emptyset$ open set G is X itself, the condition $\xi \subseteq Cl(\xi \cup int(G))$ is always satisfied because $int(G) = X$. This means that every subset ξ of X satisfies the condition for k-openness. Therefore, in this case, τ^k (the family of all k-open sets) contains all subsets of X, which is the power set of X. This is precisely the discrete topology on X.

Case 2: τ is the discrete topology: In the discrete topology, every subset of X is open. According to Definition 2.1, the k-open sets are subsets ξ of X for which $\xi \subseteq Cl(\xi \cup int(G))$ for every non-empty set G $(G \neq X)$ that is open. Since in the discrete topology every subset is open, the condition $\xi \subseteq Cl(\xi \cup int(G))$ is always satisfied because $int(G) = G$. This means that every subset ξ of X satisfies the condition for k-openness. Therefore, in this case, τ^k (the family of all k-open sets) again contains all subsets of X, which is the power set of X. This is also the discrete topology on X. In both cases, τ^k is either the indiscrete topology or the discrete topology on X, respectively.

Example2.30: Let's consider examples for both cases mentioned in Theorem 2.29

Case 1: Indiscrete (Trivial) Topology:

In the indiscrete topology on any set X, the only open sets are the empty set and the whole set X. Let's examine the k-open sets in this case. For every subset ξ of X, since there's only one non-empty open set (which is X itself), the condition $\xi \subseteq Cl(\xi \cup int(G))$ is trivially satisfied. Therefore, every subset ξ of X is k-open in the indiscrete topology. Hence, in this case, τ^k (the family of all k-open sets) is the power set of X, which is the discrete topology on X.

Case 2: Discrete Topology: In the discrete topology on any set X, every subset is open. Let's examine the k-open sets in this case. Again, for every subset ξ of X, since every subset is open, the condition $\xi \subseteq Cl(\xi \cup int(G))$ is trivially satisfied for any open set $G(G \neq X)$. Therefore, every subset ξ of X is k-open in the discrete topology. As in the previous case, τ^k (the family of all k-open sets) is the power set of X, which is the discrete topology on X. In both cases, the theorem holds true, and the family of all k-open sets coincide with either the discrete topology or the indiscrete topology on the given set X.

Lemma 2.31: If ξ is a subset of a discrete top. sp. X then the k-derived set of ξ, denoted $D_k(\xi)$, is empty.

Proof: In a discrete top. sp., every subset is open, and thus every subset is closed. This means that for any subset ξ of $X, Cl_k(\xi) = \xi$, because ξ is already closed. According to Definition 2.13, the k-derived set of ξ, denoted $D_k(\xi)$, consists of the k-limit points of ξ. By Definition 2.13, a point x is a k-limit point of ξ if for every k-open set G containing $x, G \cap (\xi\{x\}) \neq \emptyset$. In a discrete top. sp., every subset is open. So, for any point $x \in X$, we can consider the singleton set $\{x\}$ as a k-open set that contains x. Since $\{x\} \cap (\xi\{x\}) = \emptyset$ for any $x \in \xi$, this means that no point in ξ

satisfies the condition for a k-limit point. Therefore, $D_k(\xi)$ is the empty set $(\emptyset)$ for any subset ξ of a discrete top. sp.

Definition 2.32. Let X be a top. sp. and let $\xi \subseteq X$. Then $b_k(\xi) = \xi \backslash Int_k(\xi)$ is called the k-border of ξ, and the set $Fr_k(\xi) = Cl_k(\xi) \backslash Int_k(\xi)$ is called the k-frontier of ξ. Note that if ξ is a k-closed subset of X, then $b_k(\xi) = Fr_k(\xi)$.

Example 2.33: Let $X = \{a, b, c\}$ with topology $\tau = \{\emptyset, X, \{b\}, \{b, c\}\}$, then $\tau^k = \{\emptyset, X, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}\}$.

1. If $\xi = \{a, b\}$, then $Int_k(\xi) = \{a, b\}, b_k(\xi) = \xi,\ Int_k(\xi) = \emptyset, Cl_k(\xi)\ )\{b, c\}, Fr_k(\xi) = Cl_k(\xi)\ , Int_k(\xi)\ = \{c\}$.
2. if $\xi = \{b, c\}$,then $Int_k(\xi)\ ) = \{b, c\}, b_k(\xi) = \xi,\ Int_k(\xi) = \emptyset\ , Cl_k(\xi) = \{b, c\}, Fr_k(\xi)\ ) =\ Cl_k(\xi)\ , Int_k(\xi)\ =\ \emptyset$.

Theorem 2.34: If ξ is a k-closed subset of X, then the k-border of ξ, denoted $b_k(\xi)$, is equal to the k-frontier of ξ, denoted $Fr_k(\xi)$.

Proof: Let A be a k-closed subset of X. This means $\xi = Cl_k(\xi)$ by definition.

1. $b_k(\xi)$: The k-border of ξ, denoted $b_k(\xi)$, is $Int_k(\xi)$. Since ξ is k-closed, $\xi = Cl_k(\xi) = Int_k(\xi)$(since every k-closed set is k-open as well). Therefore, $b_k(\xi) = \xi \backslash \xi = \emptyset$.

2. $Fr_k(\xi)$: The k-frontier of ξ, denoted $Fr_k(\xi)$, is $Cl_k(\xi) \backslash Int_k(\xi)$. Since ξ is k-closed, $Cl_k(\xi) = \xi$, and as mentioned earlier, $\xi = Int_k(\xi)$. Therefore, $Fr_k(\xi) = Cl_k(\xi) \backslash Int_k(\xi) = \xi \backslash \xi = \emptyset$. So, $b_k(\xi) = Fr_k(\xi) = \emptyset$.

Example 2.35: Consider the top. sp.: $X = \{a, b, c\}, \tau = \{\emptyset, X, \{b\}, \{b, c\}\}$.Let's take $\xi = \{a, b, c\}$. We can see that ξ is k-closed since it's equal to $Cl_k(\xi) = \xi$. Now the k-border and k-frontier of ξ: k-border of ξ , $(b_k(\xi))$: $b_k(\xi) = \xi \backslash Int_k(\xi)$. Since ξ is k-closed, $\xi = Cl_k(\xi) = Int_k(\xi)$. Therefore, $b_k(\xi) = \xi \backslash \xi = \emptyset$. and indeed, $b_k(\ \xi) = Fr_k(\ \xi) = \emptyset$, confirming the statement.

Lemma 2.36: Let X be a top. sp. and let $\xi \subseteq X$. Then ξ is k-closed iff the k-frontier of ξ, denoted $Fr_k(\ \xi)$, is a subset of ξ.

Proof: ξ is k-closed $\Rightarrow Fr_k(\ \xi) \subseteq \xi$. Assume that ξ is k-closed. This means that $\xi = Cl_k(\ \xi)$. Now let's consider the k-frontier of $\xi, Fr_k(\ \xi)$, which is $Cl_k(\ \xi) \backslash Int_k(\ \xi)$.Since ξ is k-closed, $\xi = Cl_k(\ \xi)$, and this implies that $Cl_k(\ \xi) = Int_k(\ \xi)$. Therefore, $Fr_k(\ \xi) = Cl_k(\ \xi) \backslash Int_k(\ \xi) = \xi \backslash\ \xi = \emptyset$.An empty set is indeed a subset of any set, so $Fr_k(\ \xi) \subseteq \xi$.

$Fr_k(\ \xi) \subseteq \xi \Rightarrow \xi$ is k-closed. Assume that $Fr_k(\ \xi) \subseteq \xi$. This means that every element in $Fr_k(\ \xi)$ is also an element of ξ. We need to show that ξ is k-closed, meaning that $\xi = Cl_k(\ \xi)$. Let's consider a k-closed set that contains ξ, denoted C. We want to show that ξ is contained in C, and C is contained in ξ. Since $Fr_k(\xi) \subseteq \xi$, every point in $Fr_k(\xi)$is also in ξ. Since $Fr_k(\xi)$ is part of $Cl_k(\xi)$, and C is k-closed containing ξ, it must also contain $Fr_k(\xi)$. This implies that $C \subseteq$

ξ. On the other hand, since ξ is contained in C, $Int_k(\xi)$ is a subset of C as well. This means that $Cl_k(Int_k(\xi))$is also a subset of C. Since $\xi = Cl_k(Int_k(\xi))$ (because ξ is k-closed), this means ξ is a subset of C. Therefore, $\xi = C$, and this shows that ξ is contained in every k-closed set containing it, which means $\xi = Cl_k(\xi)$. Hence, in this direction, ξ is k-closed. Combining both directions, we conclude that ξ is k-closed iff $Fr_k(\xi) \subseteq \xi$.

Definition 2.37: Let X be a top. sp. and let $\xi \subseteq X$. Then $Ext_k(\xi) = Int_k(X\backslash\xi)$ is called the k-exterior of ξ.

Example 2.38: Given the top. sp. $X = \{a, b, c\}, \tau = \{\emptyset, X, \{a\}, \{a, b\}\}$. The family of all k-open sets, $\tau^k = \{\emptyset, \{a\}, \{a, b\}, \{b\}, \{c\}, \{a, b, c\}\}$. The $Ext_k(\xi) = \{b\}$

Theorem 2.39. Let X be a top. sp. and let $\xi \subseteq B \subseteq X$. Then

1. $Ext_k(\xi)$ is k-open.
2. $Ext_k(\xi) = X\backslash Cl_k(\xi)$.
3. If $\xi \subseteq B$, then $Ext_k(B) \subseteq Ext_k(\xi)$.
4. $Ext_k(\xi \cup B) \subseteq Ext_k(\xi) \cap Ext_k(B)$.
5. $Ext_k(\xi \cap B) \supseteq Ext_k(\xi) \cup Ext_k(B)$.
6. $Ext_k(X) = \emptyset, Ext_k(\emptyset) = X$.
7. $Ext_k(\xi) = Ext_k(X\backslash Ext_k(\xi))$.
8. $X = Int_k(\xi) \cup Ext_k(\xi) \cup Fr_k(\xi)$.

Example2.40: Given the top. sp. $X = \{a, b, c\}, \tau = \{\emptyset, X, \{a\}, \{a, b\}\}$, Let's consider $\xi = \{a, c\}$.

1. $Ext_k(\xi) = Int_k(X\backslash\xi)$. Since $X\backslash\xi = \{b\}$, and $\{b\}$ is a k-open set contained in $X\backslash\xi, Ext_k(\xi) = \{b\}$.
2. $Ext_k(\xi) = X\backslash Cl_k(\xi)$; $Cl_k(\xi) = \xi = \{a, c\}$, so $X\backslash Cl_k(\xi) = X\backslash\{a, c\} = \{b\}$.
3. If $\xi = \{a, c\}$, then $Ext_k(B) \subseteq Ext_k(\xi)$ for any subset $B \supseteq \xi$.
4. $Ext_k(\xi \cup B) \subseteq Ext_k(\xi) \cap Ext_k(B)$ for any subset $B \supseteq \xi$.
5. $Ext_k(\xi \cap B) \supseteq Ext_k(\xi) \cup Ext_k(B)$ for any subset $B \supseteq \xi$.
6. $Ext_k(X) = \emptyset, Ext_k(\emptyset) = X$.
7. $Ext_k(\xi) = Ext_k(X\backslash Ext_k(\xi))$; $Ext_k(\xi) = \{b\}$, and $X\backslash Ext_k(\xi) = \{b\}$, so $Ext_k(\xi) = Ext_k(X\backslash Ext_k(\xi))$.
8. $X = Int_k(\xi) \cup Ext_k(\xi) \cup Fr_k(\xi)$; $X = \{a, b, c\} = Int_k(\{a, c\}) \cup Ext_k(\{a, c\}) \cup Fr_k(\{a, c\})$.

Definition 2.41: A function $f : (X, \tau) \to (Y, \sigma)$ is said to be:

1. “Totally-continuous if $f^{-1}(U)$is a clopen set in X, for every open set U in Y.”
2. “contra-continuous if $f^{-1}(U)$ is a closed set in X, for every open set U in Y.”

In these definitions, $f^{-1}(U)$ denotes the preimage of the set U under the function f.

Explanation 2.42:

1. A function is totally-continuous if the preimage of every open set in the codomain Y is a clopen (both closed and open) set in the domain X.
2. A function is contra-continuous if the preimage of every open set in the codomain Y is a closed set in the domain X.

k-continuous functions and k-homeomorphism

Within this section, we present novel categories of functions referred to as k-cont. funs, k -open funs, k -irresolute funs, k -totally cont. funs, k -contra-cont. funs, k -homeomorphism and examine certain characteristics of these functions.

Definition 3.1: A fun. $f: (X, \tau) \to (Y, \sigma)$ is referred to as k-continuous if the inverse image $f^{-1}(U)$ is a k -open in X for every open U in Y.

Example3.2: Let $X = \{1, 2, 3\}, Y = \{a, b, c, d\}, \tau = \{\emptyset, X, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{2, 3\}\}, \sigma = \{\emptyset, Y, \{a, b\}, \{c, d\}\}$.Consider $f: (X, \tau) \to (Y, \sigma)$ defined as follows: $f(1) = a, f(2) = b, f(3) = c$ .Now, let's verify whether this function is k -continuous according to Definition 3.1.For every open set U in Y, let's check whether the preimage $f^{-1}(U)$ is a k -open set in X: $U = \emptyset, f^{-1}(\emptyset) = \emptyset$, which is k -open. $U = Y, f^{-1}(Y) = X$, which is k -open. $U = \{a, b\}, f^{-1}(\{a, b\}) = \{1, 2\}$, which is k -open (subset of $\{1, 2\}$). $U = \{c, d\}, f^{-1}(\{c, d\}) = \{3\}$, which is k -open. For every open set U in Y, the preimage $f^{-1}(U)$ is a k-open set in X. Therefore, the function $f: (X, \tau) \to (Y, \sigma)$ is k -continuous according to Definition 3.1 for the given top. sp. τ and σ.

Theorem 3.3: Every continuous function is k -continuous.

Proof: Let $f: (X, \tau) \to (Y, \sigma)$ be a continuous function, where X and Y are top. sp. with topologies τ and σ, respectively. To prove, for every open set U in Y, the inverse image $f^{-1}(U)$ is a k-open set in X. By the definition of a continuous function, for every open set U in Y, the preimage $f^{-1}(U)$ is an open set in X. Now we need to show that this preimage is also k-open. Consider any open set U in Y. We want to show that $f^{-1}(U)$ is k-open in X. We know that $f^{-1}(U)$ is an open set in X (because f is continuous). Now, let's consider any subset ξ of X. We need to show that $\xi \subseteq Cl(\xi \cup Int(f^{-1}(U)))$. Since $f^{-1}(U)$ is open, $Int(f^{-1}(U)) = f^{-1}(U)$ (the interior of an open set is the set itself). Therefore, we need to show that $\xi \subseteq Cl(\xi \cup f^{-1}(U))$. Since $Cl(\xi \cup f^{-1}(U))$ is the smallest closed set containing $\xi \cup f^{-1}(U)$, and $f^{-1}(U)$ is open, we have $Cl(\xi \cup f^{-1}(U)) = Cl(\xi) \cup f^{-1}(U)$. Since $f^{-1}(U)$is open, $Cl(\xi) \cup f^{-1}(U)$ is a closed set containing ξ. This implies that $\xi \subseteq Cl(\xi) \subseteq Cl(\xi) \cup f^{-1}(U)$. Therefore, for every subset ξ of X, $\xi \subseteq Cl(\xi \cup f^{-1}(U))$, which means that $f^{-1}(U)$ is k-open. Since this holds for every open set U in Y, we have shown that the preimage $f^{-1}(U)$ is k-open for every open set U in Y. Thus, f is k-continuous.

Remark 3.4: The converse of Theorem 3.3 is not necessarily true.

However, the converse of this statement, which would imply that if a function is k-continuous, it is also continuous, is not always true. In other words, there exist k- continuous function that are not continuous.

Example:3.5

Case 1: Consider the top. sp.: $X = \{a, b, c\}, \tau = \{\emptyset, X, \{a\}, \{c\}, \{a, c\}\}$,Let $Y = X$, and let the identity function be defined as $f(x) = x$ for all $x \in X$.This function is k-continuous, because for any open set U in X, $f^{-1}(U)$is the same as U, which is a k-open set in X. However, this function is not continuous. For example, consider the open set $U = \{a\}$ in X. The preimage$f^{-1}(U)$ is $\{a\}$, which is not an open set in X with respect to the given topology τ.

Case 2: Consider the same top. sp. X and τ as in the previous example. Let $Y = \{p\}$, where p is a single point, and let the constant function be defined as $f(x) = p, \forall$ x $\in$ X . This function is k-continuous, as for any open set U in Y (which is either $\emptyset$ or Y), $f^{-1}(U)$ is either $\emptyset$ or X (which is k-open). However, this function is not continuous, as the preimage of any open set U in Y (which is either $\emptyset$ or Y) is either $\emptyset$ or X (which is not open in X).

Theorem 3.6. If $f: (X, \tau) \rightarrow Y, \sigma)$ is k-continuous and $g: (Y, \sigma) \rightarrow (Z, \eta)$ is continuous, then $g \circ f : (X, \tau) \rightarrow (Z, \eta)$ is k-continuous.

Proof: Let $f: (X, \tau) \rightarrow (Y, \sigma)$ be a k-continuous function, and let $g: (Y, \sigma) \rightarrow (Z, \eta)$ be a continuous function. We want to prove that the composite function $g \circ f$ is k-continuous. By the k-continuity of f, for every open set U in Y, the preimage $f^{-1}(U)$ is a k-open set in X. By the continuity of g, for every open set V in Z, the preimage $g^{-1}(V)$ is an open set in Y. Now we want to show that the preimage $(g \circ f)^{-1}(V)$ is k-open for every open set V in Z.Notice that $(g \circ f)^{-1}(V) = f^{-1}(g^{-1}(V))$, which is the preimage of $g^{-1}(V)$ under f. Since $f^{-1}(g^{-1}(V))$ is the preimage of an open set $g^{-1}(V)$ in Y under the k-continuous function f, it is a k-open set in X. Therefore, we have shown that for every open set V in Z, the preimage $(g \circ f)^{-1}(V) = f^{-1}(g^{-1}(V))$ is k-open in X. This holds for every open set V in Z, so the composite function $g \circ f$ is k-continuous.

Definition 3.7: A function $f: (X, \tau) \rightarrow (Y, \sigma)$ is said to be k-open if for every open set U in X, the image $f(U)$ is a k-open set in Y.

Theorem 3.8: Every function that preserves openness is a k-open function.

Proof: Let $f: (X, \tau) \rightarrow (Y, \sigma)$ be an open function, where X and Y are top. sp. with topologies τ and σ, respectively. We want to show that for every open set U in X, the image $f(U)$ is a k-open set in Y. By the definition of an open function, for every open set U in X, the image $f(U)$ is an open set in Y. Now we need to show that this image $f(U)$ is also k-open. Consider any open set U in X. We want to show that $f(U)$ is k-open in Y. Since $f(U)$ is open in Y (by the assumption that f is an open function), it is also k-open, as every open set is trivially k-open. Therefore, we have shown that for every open set U in X, the image $f(U)$ is k-open in Y. This holds for every open set U in X, so we have proved that every open function is k-open.

Remark 3.9. The opposite of Theorem 3.3 does not necessarily hold, as demonstrated in the subsequent example.

Example 3.10. In Example 3.3, the function of identity$f: (X, \tau) \to (Y, \sigma)$ is k-open but not open.

Theorem 3.11: If $f: (X, \tau) \to (Y, \sigma)$ is open and $g: (Y, \sigma) \to (Z, \eta)$ is k-open, then the composite function $g \circ f: (X, \tau) \to (Z, \eta)$ is k-open.

Proof: Let $f: (X, \tau) \to (Y, \sigma)$ be an open function, and let $g: (Y, \sigma) \to (Z, \eta)$ be a k-open function. We want to prove that for every open set U in X, the image $(g \circ f)(U) = g(f(U))$ is a k-open set in Z. Since f is open, for every open set U in X, the image $f(U)$ is an open set in Y. Since g is k-open, for every open set V in Y, the image $g(V)$ is a k-open set in Z. Now, to prove, the image $(g \circ f)(U) = g(f(U))$ is k-open in Z. Consider any open set U in X. We want to show that $g(f(U))$ is k-open in Z. Since $f(U)$ is open in Y (by the assumption that f is an open function), and g is k-open, the image $g(f(U))$ is a k-open set in Z. Therefore, for every open set U in X, the image $(g \circ f)(U) = g(f(U))$ is k-open in Z. This holds for every open set U in X, so the composite function $g \circ f$ is k-open.

Definition 3.12: A function $f: (X, \tau) \to (Y, \sigma)$ is said to be k-irresolute if for every k-open set U in Y, the preimage $f^{-1}(U)$ is a k-open set in X. In simpler terms, a function is k-irresolute if the preimage of every k-open set under that function is a k-open set. This definition indicates that a k-irresolute function preserves k-openness of sets in the preimage.

Example3.13: Consider the following top. sp. and function : $X = \{1, 2, 3\}, Y = \{a, b, c\}, \tau = \{\emptyset, X, \{1\}, \{2\}, \{3\}, \{1, 2\}\}, \sigma = \{\emptyset, Y, \{a\}, \{b\}, \{c\}, \{a, b\}\}$Let's define a function $f: (X, \tau) \to (Y, \sigma)$ as follows: $f(1) = a, f(2) = b, f(3) = c$. We will now determine whether the function f is k-irresolute according to Definition 3.12. For every k-open set U in Y, we need to check whether the preimage $f^{-1}(U)$ is k-open in X. Let's consider the possible k-open sets in Y: $U = \emptyset, f^{-1}(\emptyset) = \emptyset$, which is k-open. $U = Y, f^{-1}(Y) = X$, which is k-open. $U = \{a\}, f^{-1}(\{a\}) = \{1\}$, which is k-open. $U = \{b\}, f^{-1}(\{b\}) = \{2\}$,which is k-open. $U = \{c\}, f^{-1}(\{c\}) = \{3\}$, which is k-open. $U = \{a, b\}, f^{-1}(\{a, b\}) = \{1, 2\}$, which is k-open. For every k-open set U in Y, the preimage $f^{-1}(U)$ is k-open in X. Therefore, the function $f: (X, \tau) \to (Y, \sigma)$ is k-irresolute according to Definition 3.13 for the given top. sp. τ and σ.

Theorem 3.14: Every continuous function is k-irresolute.

Proof: Let $f: (X, \tau) \to (Y, \sigma)$ be a continuous function, where X and Y are top. sp. with topologies τ and σ, respectively. We want to prove that for every k-open set U in Y, the preimage $f^{-1}(U)$ is a k-open set in X. By the continuity of f, for every open set U in Y, the preimage $f^{-1}(U)$ is an open set in X. Now, let's consider a k-open set U in Y. Since every k-open set is open, U is also open. Therefore, the preimage $f^{-1}(U)$, which is open due to the continuity of f, is also a k-open set. This holds for every k-open set U in Y. Hence, $f^{-1}(U)$ is k-open for every k-open set U in Y. Thus, we have proved that every continuous function is k-irresolute.

Theorem 3.15: Every k-irresolute function is k-continuous.

Proof: Let $f:(X,\tau)\to(Y,\sigma)$ be a k-irresolute function, where X and Y are top. sp. with topologies τ and σ, respectively. We want to prove that for every open set U in Y, the preimage $f^{-1}(U)$ is k-open in X. By the definition of a k-irresolute function, for every k-open set U in Y, the preimage $f^{-1}(U)$ is k-open in X. Now, consider an open set U in Y. We want to show that the preimage $f^{-1}(U)$ is k-open in X. Since every open set is also k-open, the preimage $f^{-1}(U))$ is k-open in X due to the k-irresolute property of f. This holds for every open set U in Y. Hence, the preimage $f^{-1}(U)$ is k-open for every open set U in Y. Therefore, we have proved that every k-irresolute function is k-continuous.

Theorem 3.16: The composition of two k-irresolute function is also k-irresolute.

Proof: Let $f:(X,\tau)\to(Y,\sigma)$ and $g:(Y,\sigma)\to(Z,\eta)$ be two k-irresolute functions, where X, Y, and Z are top. sp. with topologies τ, σ, and η, respectively. To prove that the composition $g\circ f:(X,\tau)\to(Z,\eta)$ is k-irresolute. By the k-irresolute property of f, for every k-open set U in Y, the preimage $f^{-1}(U)$ is k-open in X. Similarly, by the k-irresolute property of g, for every k-open set V in Z, the preimage $g^{-1}(V)$ is k-open in Y. Now, consider a k-open set V in Z. We want to show that the preimage $(g\circ f)^{-1}(V)$ is k-open in X. Since g is k-irresolute, the preimage $g^{-1}(V)$ is k-open in Y. Since f is k-irresolute, the preimage $f^{-1}(g^{-1}(V))$ is k-open in X. However, the preimage $(gf)^{-1}(V)$ is precisely $f^{-1}(g^{-1}(V))$. Thus, the preimage$(g\circ f)^{-1}(V)$ is k-open in X. This holds for every k-open set V in Z. Hence, the preimage $(g\circ f)^{-1}(V)$ is k-open for every k-open set V in Z. Therefore, we have proved that the composition $g \circ f$ is k-irresolute.

Theorem 3.17: If $f:(X,\tau)\to(Y,\sigma)$ is k-irresolute and $g:(Y,\sigma)\to(Z,\eta)$ is k-continuous, then the composition $g\circ f:(X,\tau)\to(Z,\eta)$ is k-irresolute.

Proof: Let $f:(X,\tau)\to(Y,\sigma)$ be a k-irresolute function, and $g:(Y,\sigma)\to(Z,\eta)$ be a k-continuous function. To prove that the composition $g\circ f:(X,\tau)\to(Z,\eta)$ is k-irresolute. By the k-irresolute property of f, for every k-open set U in Y, the preimage $f^{-1}(U)$ is k-open in X. By the k-continuous property of g, for every k-open set V in Z, the preimage $g^{-1}(V)$ is k-open in Y. Now, consider a k-open set V in Z. We want to show that the preimage $(g\circ f)^{-1}(V)$ is k-open in X. Since g is k-continuous, the preimage $g^{-1}(V)$ is k-open in Y. Since f is k-irresolute, the preimage $f^{-1}(g^{-1}(V))$ is k-open in X. However, the preimage $(g\circ f)^{-1}(V)$ is precisely $f^{-1}(g^{-1}(V))$. Thus, the preimage $(g\circ f)^{-1}(V)$ is k-open in X. This holds for every k-open set V in Z. Hence, the preimage $(g\circ f)^{-1}(V)$ is k-open for every k-open set V in Z. Therefore, we have proved that the composition $g\circ f$ is k-irresolute.

Definition 3.18: A bijective function $f: (X, \tau) \to (Y, \sigma)$is said to be a k-homeomorphism if f is both k-continuous and k-open. In simpler terms, a k-homeomorphism is a bijective function between top. sp. that preserves the k-openness and k-continuity properties.

Theorem 3.19. If $f: (X, \tau) \to (Y, \sigma)$is homomorphism, then f is k-homomorphism

Proof: Let $f: (X, \tau) \to (Y, \sigma)$be a homeomorphism between the top. sp. X and Y. To show that f is also a k-homeomorphism. Since f is a homeomorphism, it is both continuous and open. This means that for every open set U in X, the image $f(U)$ is open in Y. Similarly, for every open set V in Y, the preimage $f^{-1}(V)$ is open in X. We will now prove that f is k-continuous and k-open.

K-Continuity: Consider a k-open set U in Y. We want to show that the preimage $f^{-1}(U)$ is k-open in X. Since f is continuous, the preimage $f^{-1}(U)$ is open in X. Since every open set is also k-open, the preimage $f^{-1}(U)$ is k-open in X. Therefore, f is k-continuous.

K-Openness: Consider a k-open set V in X. To show that the image $f(V)$ is k-open in Y. Since f is open, the image$f(V)$ is open in Y. Since every open set is also k-open, the image $f(V)$ is k-open in Y. Therefore, f is k-open. Since f is both k-continuous and k-open, it is a k-homeomorphism.

Remark 3.20. The converse of the Theorem 3.19, need not be true as shown in the following example. **Example 3.21.** Let $X = Y = \{a, b, c\}, \tau = \{\emptyset, X, \{a, c\}\}$, $\sigma = \{\emptyset, Y, \{b, c\}\}$ .Then $\tau^k = \{\emptyset, \{a\}, \{c\}, \{a, c\}\}$ and $\sigma^k = \{\emptyset, \{b, c\}\}$. Now, let's consider the function $f: (X, \tau) \to (Y, \sigma)$ defined as follows:$f(a) = b$, $f(b) = c, f(c) = b$,We will show that f is a k-homeomorphism, but it is not a homeomorphism.

K-Continuity and K-Openness: For every k-open set U in Y, the preimage $f^{-1}(U)$ is k-open in X due to the structure of the topologies and the function f. Similarly, for every k-open set V in X, the image $f(V)$ is k-open in Y. However, f is not a homeomorphism because it is not a bijection. Specifically, both b and c in Y are mapped to b in X by f, which makes f not bijective.

Definition 3.22. A function $f: (X, \tau) \to (Y, \sigma)$is said to be k-totally continuous, if $f^{-1}(U)$ is clopen set in X for every k-open set U in Y.

Theorem 3.23: Every k-totally continuous function is totally continuous.

Proof: Let $f: (X, \tau) \to (Y, \sigma)$be a k-totally continuous function. To show that f is totally continuous. Recall that a function is totally continuous if the preimage of every open set in Y is a clopen set in X. Let U be an open set in Y. We want to show that the preimage $f^{-1}(U)$ is clopen in X. Since f is k-totally continuous, the preimage $f^{-1}(U)$ is clopen for every k-open set U in Y. Since every open set in Y is also a k-open set, this implies that the preimage $f^{-1}(U)$ is clopen for every open set U in Y. Therefore, f is totally continuous.

Theorem 3.24: Every k-totally continuous function is k-irresolute.

Proof: Let $f: (X, \tau) \to (Y, \sigma)$be a k-totally continuous function. We want to show that f is k-irresolute. Recall that a function is k-irresolute if the preimage of every k-open set in Y is a k-open set in X. Let U be a k-open set in Y. We want to show that the preimage $f^{-1}(U)$ is k-open in X. Since f is k-totally continuous, the preimage $f^{-1}(U)$ is clopen for every k-open set U in Y. Since every k-open set is also an open set, this implies that the preimage $f^{-1}(U)$ is clopen for every open set U in Y. And since every open set is both closed and open, it follows that the preimage $f^{-1}(U)$ is both closed and open in X. Since every k-open set is also open, the preimage $f^{-1}(U)$ is k-open in X. Therefore, f is k-irresolute.

Remark 3.25: The converse of the Theorem 3.24, need not be true as shown in the following example. **Example 3.26:** In Example Let $X = \{a, b, c\}$ and $Y = \{1, 2, 3\}, \tau = \{\emptyset, X, \{b\}\}, \sigma = \{\emptyset, Y, \{1\}, \{2, 3\}\}$. A function $f: (X, \tau) \to (Y, \sigma)$is defined by $f(\{a\}) = \{2\}, f(\{b\}) = \{1\}, f(\{c\}) = \{3\}$., the identity function f : (X, τ) → (Y, σ) is k-irresolute but not k-totally continuous. Consider the following top. sp. and function, $\tau^k = \{\emptyset, \{b\}\}, \sigma^k = \{\emptyset, \{1\}, \{2, 3\}\}$. We will analyze whether f is k-irresolute and k-totally continuous.

k-Irresoluteness: For every k-open set U in Y, the preimage $f^{-1}(U)$ is k-open in X due to the structure of the function f. However, f is not an open function, which means it is not k-irresolute.

K-Totally Continuous: We will analyze the function $f^{-1}(\{1\})$. The preimage $f^{-1}(\{1\}) = \{b\}$, which is a clopen set in X. However, $\{b\}$ is not a k-open set in X since it doesn't satisfy the requirement for k-openness (it should be contained in the closure of the union of a k-open set and X). Therefore, the identity function $f: (X, \tau) \to (Y, \sigma)$is k-irresolute but not k-totally continuous.

Theorem 3.27: The composition of two k-totally continuous functions is also k-totally continuous.

Proof: Let $f: (X, \tau) \to (Y, \sigma)$and $g: (Y, \sigma) \to (Z, \eta)$ be two k-totally continuous functions. We want to show that the composition $g \circ f: (X, \tau) \to (Z, \eta)$ is also k-totally continuous. Recall that a function is k-totally continuous if the preimage of every k-open set is clopen. Let U be a k-open set in Z. We want to show that the preimage $(g \circ f)^{-1}(U)$ is clopen in X. Since g is k-totally continuous, the preimage $g^{-1}(U)$ is clopen for every k-open set U in Z. Similarly, since f is k-totally continuous, the preimage $f^{-1}(g^{-1}(U))$ is clopen for every k-open set U in Z. Now, notice that $(g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U))$. Since the preimage $f^{-1}(g^{-1}(U))$ is clopen for every k-open set U in Z, this implies that $(g \circ f)^{-1}(U)$ is clopen for every k-open set U in Z. Therefore, the composition $g \circ f$ is k-totally continuous.

Theorem 3.28: If $f: (X, \tau) \to (Y, \sigma)$is k-totally continuous and $g : (Y, \sigma) \to (Z, \eta)$ is k-irresolute, then the composition $g \circ f : (X, \tau) \to (Z, \eta)$ is k-totally continuous.

Proof: Let $f: (X, \tau) \to (Y, \sigma)$be a k-totally continuous function and $g: (Y, \sigma) \to (Z, \eta)$ be a k-irresolute function. We want to show that the composition $g \circ f$ is k-totally continuous. Recall that a function is k-totally continuous if the preimage of every k-open set is clopen. Let U be a k-open set in Z. We want to show that the preimage $(g \circ f)^{-1}(U)$ is clopen in X. Since g is k-

irresolute, the preimage $g^{-1}(U)$ is k-open in Y for every k-open set U in Z. Also, since f is k-totally continuous, the preimage $f^{-1}(g^{-1}(U))$ is clopen in X for every k-open set U in Z. Now, notice that $(g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U))$. Since the preimage $f^{-1}(g^{-1}(U))$ is clopen for every k-open set U in Z, this implies that $(g \circ f)^{-1}(U)$ is clopen for every k-open set U in Z. Therefore, the composition $g \circ f$ is k-totally continuous.

Theorem 3.29: If $f:(X,\tau) \to (Y,\sigma)$is k-totally continuous and $g:(Y,\sigma) \to (Z,\eta)$ is k-continuous, then the composition $g \circ f:(X,\tau) \to (Z,\eta)$ is totally continuous.

Proof: Let $f:(X,\tau) \to (Y,\sigma)$ be a k-totally continuous function and $g:(Y,\sigma) \to (Z,\eta)$ be a k-continuous function. We want to show that the composition $g \circ f$ is totally continuous. Recall that a function is totally continuous if the preimage of every open set is a clopen set. Let U be an open set in Z. We want to show that the preimage $(g \circ f)^{-1}(U)$ is clopen in X. Since g is k-continuous, the preimage $g^{-1}(U)$ is k-open in Y for every open set U in Z. Also, since f is k-totally continuous, the preimage $f^{-1}(g^{-1}(U))$ is clopen in X for every open set U in Z. Now, notice that $(g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U))$.. Since the preimage$f^{-1}(g^{-1}(U))$. is clopen for every open set U in Z, this implies that $(g \circ f)^{-1}(U)$ is clopen for every open set U in Z. Therefore, the composition $g \circ f$ is totally continuous.

Definition 3.30: A function $f:(X,\tau) \to (Y,\sigma)$is said to be k-contra-continuous if the preimage $f^{-1}(U)$ is k-closed in X for every open set U in Y. In other words, a function is k-contra-continuous if the inverse image of every open set in the codomain is a k-closed set in the domain.

Theorem 3.31: Every contra-continuous function is k-contra-continuous.

Proof: Let $f:(X,\tau) \to (Y,\sigma)$ be a contra-continuous function. We want to show that f is also k-contra-continuous. Recall that a function is contra-continuous if the preimage of every open set is closed. A function is k-contra-continuous if the preimage of every open set is k-closed. Let U be an open set in Y. Since f is contra-continuous, the preimage $f^{-1}(U)$ is closed in X for every open set U in Y. Now, since closed sets are a subset of k-closed sets, it follows that the preimage $f^{-1}(U)$ is also k-closed in X for every open set U in Y. Therefore, the function f is k-contra-continuous.

Theorem 3.32: Every totally continuous function is k-contra-continuous.

Proof: Let $f:(X,\tau) \to (Y,\sigma)$ be a totally continuous function. We want to show that f is also k-contra-continuous. Recall that a function is totally continuous if the preimage of every open set is clopen. A function is k-contra-continuous if the preimage of every open set is k-closed. Let U be an open set in Y. Since f is totally continuous, the preimage $f^{-1}(U)$ is clopen in X for every open set U in Y. Since closed sets are a subset of k-closed sets, it follows that the preimage $f^{-1}(U)$ is also k-closed in X for every open set U in Y. Therefore, the function f is k-contra-continuous.

Remark and example 3.33: The converse of Theorem 3.32. Consider the following top. sp. and function: $X = Y = \{a,b,c\}, \tau = \{\emptyset, X, \{b\}, \{b,c\}\}, \sigma = \{\emptyset, Y, \{b\}\}, \tau^k = \{\emptyset, \{b\}, \{b,c\}\}, \sigma^k = \{\emptyset, \{b\}\}$. Now let's analyze the given identity function $f:(X,\tau) \to (Y,\sigma)$:The identity function is defined as: $f(x) = x$ for all $x \in X$. We want to determine whether the identity function is k-contra-continuous but not totally continuous. Recall that a function is k-contra-continuous if the preimage of every open set is k-closed. Let U be an open set in Y. Since the open sets in Y are $\{\emptyset, \{b\}\}$, we need to consider the preimages $f^{-1}(\emptyset)$ and $f^{-1}(\{b\})$. $f^{-1}(\emptyset) = \emptyset$, which is k-closed. $f^{-1}(\{b\}) = \{b\}$, which is also k-closed. Since the preimages of all open sets are k-closed, the identity function is k-contra-continuous. However, the identity function maps open sets to open sets, and closed sets to closed sets, preserving the topology. Therefore, it is also totally continuous.

CONCLUSION

The exploration of k-open sets and their associated concepts has provided a deeper understanding of the intricate relationships and properties within top. sp.. The journey began with the introduction of k-open sets and the unveiling of their unique characteristics. This led to the unveiling of a plethora of concepts, such as k-interior, k-closure, k-limit points, k-derived sets, k-borders, k-frontiers, and k-exteriors. Each of these concepts added a layer of complexity and richness to the study of topology.

Throughout the investigation, a variety of theorems and propositions were established, revealing the interconnectedness of these concepts. Notably, the equivalence of certain properties, such as the relationship between k-limit points and k-closures, provided a deeper insight into the nature of top. sp.. The introduction of k-functions, including k-continuous, k-irresolute, and k-totally continuous functions, expanded the exploration into the realm of functions and their interactions with k-open sets.

It was clear from the examples and counterexamples provided that not all properties held universally. The exploration of these counterexamples emphasized the importance of understanding the specific conditions and contexts in which certain relationships and properties are valid. This added a layer of nuance to the study, reminding us that topology is a field rich in intricacies and exceptions.

The journey through k-open sets and related concepts ultimately contributed to a broader appreciation of the depth and complexity of topology. The concrete examples, rigorous definitions, and intricate theorems illuminated the landscape of top. sp., providing a fresh perspective on how subsets, functions, and open sets interact. This exploration serves as a reminder that the world of topology is both fascinating and challenging, inviting further investigation and discovery.

In conclusion, the study of k-open sets and their properties has enriched our understanding of topology and its various facets. The concepts, theorems, and examples presented have provided a comprehensive view of the intricate relationships that govern top. sp., setting the stage for further exploration and inquiry into the world of mathematics.

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